JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Journal of Computational and Applied Mathematics 114 (2000) 405–409

Letter to the Editor

A modification of the super-Halley method under mild differentiability conditions ☆

J.A. Ezquerro *, M.A. Hernández

Department of Mathematics and Computation, University of La Rioja, C/Luis de Ulloa s/n, 26004–Logroño, Spain

Received 18 May 1999

Abstract

A new two-point iteration of order three is introduced to approximate a solution of a nonlinear operator equation in Banach spaces. Under the same assumptions as for Newton's method, we provide a result on the existence of a unique solution for the nonlinear equation, which is based on a technique consisting of a new system of recurrence relations.

MSC: 47H17; 65J15

Keywords: Nonlinear equations in Banach spaces; Two-point iteration; Convergence theorem; Recurrence relations

1. Introduction

We consider the problem of solving

$$F(x) = 0, \tag{1}$$

where $F : \Omega \subseteq X \to Y$ is a nonlinear once Fréchet differentiable operator in an open convex domain Ω of a Banach space X with values in a Banach space Y.

Multipoint methods are defined as iterations which use new information at a number of points. A very restrictive condition on one-point iterations of order N is that they depend explicitly on the first $N-1$ derivatives of F. This implies that their informational efficiency is less than or equal to unity. Neither of these restrictions need hold for multipoint methods, that is, for iterations which sample F and its derivatives at a number of values of the independent variable.

☆ The research reported herein was sponsored in part by the University of La Rioja (grant API-99/B14).

* Corresponding author. Tel.: +34-941-299440; fax: +34-941-299460.

E-mail addresses: jezquer@dmc.unirioja.es (J.A. Ezquerro), mahernan@dmc.unirioja.es (M.A. Hernández)

PII: S0377-0427(99)00348-9

We derive a new family of two-step methods from one of the most famous one-point iterations of order three, called the convex acceleration of Newton's method or the super-Halley method [2,4]:

$$G(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x),$$

$$x_{n+1} = x_n - [I + \tfrac{1}{2}G(x_n)(I - G(x_n))^{-1}]F'(x_n)^{-1}F(x_n), \quad n \geqslant 0.$$

From Taylor's formula, we have

$$F'(z_n) \approx F'(x_n) + F''(x_n)(z_n - x_n),$$

where $z_n = x_n + p(y_n - x_n)$ and $p \in (0,1]$. We can now approximate

$$F''(x_n)(y_n - x_n) \approx \frac{1}{p}[F'(z_n) - F'(x_n)]$$

and derive the two-point iteration function:

$$y_n = x_n - F'(x_n)^{-1}F(x_n),$$

$$H(x_n, y_n) = \frac{1}{p}F'(x_n)^{-1}[F'(x_n + p(y_n - x_n)) - F'(x_n)],$$

$$x_{n+1} = y_n - \tfrac{1}{2}H(x_n, y_n)[I + H(x_n, y_n)]^{-1}(y_n - x_n), \quad n \geqslant 0, \tag{2}$$

where $p \in (0,1]$, to approximate a zero x^* of (1).

Note that it is easy to check in the real case that iteration (2) is of order three at least. Moreover, for the special choice of $p = \frac{2}{3}$, we obtain the Jarratt method whose order of convergence is four (see [1]).

It is known [3,5] that for third-order iterations, necessary conditions for the convergence of (2) can be established by assuming that the second Fréchet derivative of F'' is bounded and satisfies a Lipschitz condition of the form:

$$\| F''(x) - F''(y) \| \leqslant L \| x - y \|, \quad L \geqslant 0, \quad x, y \in \Omega. \tag{3}$$

The aim of this paper is to prove the convergence of (2) by assuming only that F' satisfies a Lipschitz condition together with a point condition. That is, under the same assumptions as for Newton's method (see [1]), we provide a theorem on the existence of a unique solution for (1), which is based on a technique consisting of a new system of recurrence relations. We also show an example where our conditions are fulfilled and (3) fails.

We denote $\overline{B(x,r)} = \{y \in X; \| y - x \| \leqslant r\}$ and $B(x,r) = \{y \in X; \| y - x \| < r\}$.

2. Existence of a unique solution

Theorem 2.1. *Let F be a continuously Fréchet differentiable operator in an open convex domain $\Omega \subseteq X$. Let $x_0 \in \Omega$ be a point where the operator $\Gamma_0 = F'(x_0)^{-1}$ exists and*

$$\| \Gamma_0 \| \leqslant \beta, \quad \| y_0 - x_0 \| \leqslant \eta, \quad \| F'(x) - F'(y) \| \leqslant K \| x - y \|, \quad x, y \in \Omega.$$

We define $a_n = a_{n-1} f(a_{n-1})^2 g(a_{n-1})$ *with* $a_0 = K\beta\eta$, $f(x) = 2(1-x)/(x^2 - 4x + 2)$ *and* $g(x) = x(x^2 - 8x + 8)/8(1-x)^2$. *If we now denote*

$$R = \left(1 + \frac{a_0}{2(1-a_0)}\right)\frac{1}{1-\Delta} \quad \text{with } \Delta = 1/f(a_0)$$

and

$$a_0 < r = 0.292246\ldots \tag{4}$$

(r *is the smallest positive root of the polynomial* $q(x) = 2x^4 - 17x^3 + 48x^2 - 40x + 8$), *and* $\overline{B(x_0, R\eta)}$ *is contained in* Ω, *then sequence* (2) *is well defined for* $p \in (0,1]$, *it is contained in* $\overline{B(x_0, R\eta)}$ *and converges to a solution* x^* *of* (1). *Furthermore,* x^* *is the unique solution of* (1) *in* $B(x_0, (2/K\beta) - R\eta) \cap \Omega$.

Proof. Note that the above hypotheses guarantee the existence of x_1 in (2) and

$$\| x_1 - x_0 \| \leqslant \left(1 + \frac{a_0}{2(1-a_0)}\right) \| y_0 - x_0 \| .$$

In addition, it can be shown without difficulty that $\Gamma_1 F'(x_0)$ exists and

$$\| \Gamma_1 \| \leqslant f(a_0) \| \Gamma_0 \| .$$

Then x_2 is defined. Taking into account (2), we have

$$\Gamma_0 F(x_1) = -\frac{1}{2} H(x_0, y_0)[I + H(x_0, y_0)]^{-1}(y_0 - x_0) + \int_{x_0}^{x_1} \Gamma_0[F'(x) - F'(x_0)]\,\mathrm{d}x$$

by Taylor's formula. So

$$\| y_1 - x_1 \| \leqslant f(a_0) g(a_0) \| y_0 - x_0 \| .$$

Furthermore,

$$K \| \Gamma_1 \| \| y_1 - x_1 \| \leqslant a_1 \quad \text{and} \quad \| x_2 - x_1 \| \leqslant \left(1 + \frac{a_1}{2(1-a_1)}\right) \| y_1 - x_1 \| .$$

Finally, from (4) and $f(a_0)^2 g(a_0) < 1$ we have

$$a_1 = a_0 f(a_0)^2 g(a_0) < a_0.$$

Following an inductive procedure we can replace x_1 by x_2, x_2 by x_3 and, in general, x_{n-1} by x_n to confirm that Γ_n exists and the following recurrence relations:

$$\| \Gamma_n \| \leqslant f(a_{n-1}) \| \Gamma_{n-1} \|,$$

$$\| y_n - x_n \| \leqslant f(a_{n-1}) g(a_{n-1}) \| y_{n-1} - x_{n-1} \|,$$

$$\| H(x_n, y_n) \| \leqslant K \| \Gamma_n \| \| y_n - x_n \| \leqslant a_n,$$

$$\| x_{n+1} - x_n \| \leqslant \left(1 + \frac{a_n}{2(1-a_n)}\right) \| y_n - x_n \|,$$

$$a_n < a_{n-1}.$$

We can now prove that $\{x_n\}$ is a Cauchy sequence. Observe that

$$\left(1+\frac{a_n}{2(1-a_n)}\right)\parallel y_n-x_n\parallel \leqslant \left(1+\frac{a_0}{2(1-a_0)}\right) f(a_{n-1})g(a_{n-1})\parallel y_{n-1}-x_{n-1}\parallel$$
$$\leqslant\cdots\leqslant\left(1+\frac{a_0}{2(1-a_0)}\right)\parallel y_0-x_0\parallel\prod_{j=0}^{n-1} f(a_j)g(a_j).$$

We now have

$$\prod_{j=0}^{n-1} f(a_j)g(a_j)\leqslant\prod_{j=0}^{n-1}(\gamma^{2^j}\Delta)=\gamma^{(2^n-1)/2}\Delta^n$$

where $\gamma=a_1/a_0<1$ and $\Delta=1/f(a_0)<1$. So

$$\begin{aligned}\parallel x_{n+m}-x_n\parallel &\leqslant \parallel x_{n+m}-x_{n+m-1}\parallel+\parallel x_{n+m-1}-x_{n+m-2}\parallel+\cdots+\parallel x_{n+1}-x_n\parallel\\
&\leqslant\left(1+\frac{a_{n+m-1}}{2(1-a_{n+m-1})}\right)\eta\prod_{j=0}^{n+m-2} f(a_j)g(a_j)+\cdots\\
&\quad+\left(1+\frac{a_n}{2(1-a_n)}\right)\eta\prod_{j=0}^{n-1} f(a_j)g(a_j)\\
&\leqslant\left(1+\frac{a_n}{2(1-a_n)}\right)(\gamma^{(2^{n+m-1}-1)/2}\Delta^{n+m-1}+\cdots+\gamma^{(2^n-1)/2}\Delta^n)\eta\\
&\leqslant\left(1+\frac{a_0}{2(1-a_0)}\gamma^{(2^n-1)/2}\right)\gamma^{(2^n-1)/2}\,\frac{\Delta^n(1-\Delta^m)}{1-\Delta}\eta.\end{aligned}\tag{5}$$

For $n=0$, we obtain

$$\parallel x_m-x_0\parallel\ <\ \left(1+\frac{a_0}{2(1-a_0)}\right)\frac{(1-\Delta^m)}{1-\Delta}\eta<R\eta.$$

Similarly, we have $y_n\in B(x_0,R\eta)$ for all $n\geqslant 0$.

To see that x^* is a solution of $F(x)=0$, we have $\parallel\Gamma_nF(x_n)\parallel\to 0$ as $n\to\infty$. Taking into account that $\parallel F(x_n)\parallel\ \leqslant\ \parallel F'(x_n)\parallel\parallel\Gamma_nF(x_n)\parallel$ and the sequence $\{\parallel F'(x_n)\parallel\}$ is bounded, we infer that $\parallel F(x_n)\parallel\to 0$ as $n\to\infty$. We therefore obtain $F(x^*)=0$ by the continuity of F.

To prove uniqueness, assume some other solution y^* of (1) in $B(x_0,(2/K\beta)-R\eta)\cap\Omega$. From the approximation

$$0=\Gamma_0(F(y^*)-F(x^*))=\int_0^1\Gamma_0F'(x^*+\ t(y^*-x^*))\,\mathrm{d}t(y^*-x^*),$$

we have to prove that the operator $P=\int_0^1F'(x^*+\ t(y^*-x^*))\,\mathrm{d}t$ is invertible and then $y^*=x^*$. Indeed, from

$$\begin{aligned}\parallel\Gamma_0\parallel\int_0^1\parallel F'(x^*+t(y^*-x^*))-F'(x_0)\parallel\ \mathrm{d}t\ &\leqslant K\beta\int_0^1\parallel x^*+t(y^*-x^*)-x_0\parallel\ \mathrm{d}t\\
&\leqslant K\beta\int_0^1((1-t)\parallel x^*-x_0\parallel+t\parallel y^*-x_0\parallel)\,\mathrm{d}t\\
&<\frac{K\beta}{2}\left(R\eta+\frac{2}{K\beta}-R\eta\right)=1,\end{aligned}$$

it follows that P^{-1} exists. $\square$

We now give an example where the conditions of Theorem 2.1 are satisfied but condition (3) is not.

Example. Let us consider the system of equations $F(x,y)=0$ where $F\colon [-1,1]\times[-1,1]\to\mathbb{R}^2$ such that

$$F(x,y)=(x^3\ln x^2+2y-\tfrac{1}{16},\ x(y-2)).$$

If we choose $\boldsymbol{x}_0=(0,0)$, we observe that F does not satisfy condition (3).

On the other hand, we can apply Theorem 2.1, since

$$\beta=\parallel \Gamma_0 \parallel=\tfrac{1}{2},\quad \eta=\parallel \boldsymbol{y}_0-\boldsymbol{x}_0 \parallel=\tfrac{1}{32},\quad K=10$$

and consequently, $a_0=K\beta\eta=0.15625<r=0.292246\ldots$. As a result, we can only study the convergence for this system of equations under the hypotheses of Theorem 2.1.

References

[1] I.K. Argyros, D. Chen, Q. Qian, The Jarratt method in Banach space setting, J. Comput. Appl. Math. 51 (1994) 103–106.

[2] D. Chen, I.K. Argyros, Q. Qian, A local convergence theorem for the super-Halley method in a Banach space, Appl. Math. Lett. 7 (5) (1994) 49–52.

[3] J.M. Gutiérrez, M.A. Hernández, A family of Chebyshev–Halley type methods in Banach spaces, Bull. Austral. Math. Soc. 55 (1997) 113–130.

[4] M.A. Hernández, Newton–Raphson's method and convexity, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 22 (1) (1992) 159–166.

[5] M.A. Hernández, Reduced recurrence relations for the Chebyshev method, J. Optim. Theory Appl. 98 (1998) 385–397.